Soliton Solutions, Pairwise Collisions and Partially Coherent Interactions for a Generalized (1+1)-Dimensional Coupled Nonlinear Schrödinger System via Symbolic Computation

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With the aid of symbolic computation, we investigate a generalized (1+1)-dimensional coupled nonlinear Schrödinger system with mixed nonlinear interactions, which has potential applications in nonlinear optics and elastic solids. The exact analytical one-, two-, and three-soliton solutions are firstly obtained by employing the bilinear method under two constraints. Some main propagation and interaction properties of the solitons are discussed simultaneously. Moreover, some figures are plotted to graphically analyze the pairwise collisions and partially coherent interactions of three solitons.

Key words: Soliton Solution; Coupled Nonlinear Schrödinger System; Pairwise Collision; Partially Coherent Interaction; Symbolic Computation.

1. Introduction

Many nonlinear phenomena can be described or modeled by nonlinear evolution equations (NLEEs) [1 – 4]. One typical NLEE is the nonlinear Schrödinger (NLS) equation, a canonical equation describing the evolution of the slowly varying amplitude of a wave propagating in weakly nonlinear and dispersive media [1]. However, if there exist two or more waves in those media, the coupled NLS equations would be the relevant model [5]. Such coupled NLS models with physical interests have been widely studied in nonlinear wave motion in hydrodynamics, nonlinear optics, and even in solids [6, 7]. In [7], considering the propagation of high frequency transverse waves in a weakly nonlinear and dispersive elastic solid medium, a generalized (1+1)-dimensional coupled NLS system is obtained as

$$\begin{aligned} &\mathrm{i}\phi_\tau+p\phi_{\xi\xi}+\delta_1|\phi|^2\phi+\delta_2\psi^2\phi^*+\delta_3|\psi|^2\phi=0,\\ &\mathrm{i}\psi_\tau+p\psi_{\xi\xi}+\delta_1|\psi|^2\psi+\delta_2\phi^2\psi^*+\delta_3|\phi|^2\psi=0, \end{aligned} \tag{1}$$

where $\phi(\xi,\tau)$ and $\psi(\xi,\tau)$ are the complex amplitudes, the variables ξ and τ correspond to the normalized distance and time, respectively, the asterisk denotes the complex conjugate, and $\delta_1 \neq 0$, δ_2, $\delta_3 \neq 0$ and p are real constants describing the characteristics of the medium and of the interaction. The parameter p signifies the relative sign of the group velocity dispersion (GVD) [1], that is, a positive parameter p corresponds to the waves traveling in an anomalous GVD region, and a negative p corresponds to waves in a normal GVD region (without loss of generality). The cases, where the signs of the δ_j $(j=1,2,3)$ are not the same, i. e., the so-called "mixed" interaction type [8], are more intriguing and may have important applications in nonlinear phenomena.

Obviously, system (1) contains the following three currently important models arising from the optical fiber system (where ξ and τ denote the transverse coordinate and the coordinate along the direction of propagation):

- The integrable coupled NLS equations of Manakov type take the forms

$$\begin{aligned} &\mathrm{i}\phi_\tau+\phi_{\xi\xi}+\delta(|\phi|^2+|\psi|^2)\phi=0,\\ &\mathrm{i}\psi_\tau+\psi_{\xi\xi}+\delta(|\psi|^2+|\phi|^2)\psi=0, \end{aligned} \tag{2}$$

which govern the simultaneous propagations of two orthogonal components of an electric field in optical fibers [9] and are also important in describing the effects of averaged random birefringence on an orthogonally polarized pulse in fibers [10]. For such a system, the exact soliton solutions, elastic and inelastic collisions, together with the properties of solitons during collision including shape-changing intensity redistributions, amplitude-dependent phase shifts, and relative separation distances have been stated in detail [11]. Accordingly, in the present paper, we consider two other cases.

• The system of coupled NLS equations in the form

$$\begin{aligned} &\mathrm{i}\phi_\tau + \phi_{\xi\xi} + \delta(|\phi|^2 + \sigma|\psi|^2)\phi = 0,\\ &\mathrm{i}\psi_\tau + \psi_{\xi\xi} + \delta(|\psi|^2 + \sigma|\phi|^2)\psi = 0, \end{aligned} \tag{3}$$

describes the interactions of waves with different frequencies and orthogonally polarized components in fibers [12], the propagations and nonlinear modulations of two electromagnetic waves with almost equal group velocities in birefringent fibers [13], and also plays an important role in the theory of soliton wavelength division multiplexing [14]. As presented in [15], system (3) does not pass the Painlevé test unless $\sigma = 1$.

• The integrable coupled NLS equations which have application for the propagations of pulses in an isotropic medium are written as [16, 17]

$$\begin{aligned} &\mathrm{i}\phi_\tau + \phi_{\xi\xi} + 2(|\phi|^2 + 2|\psi|^2)\phi - 2\psi^2\phi^* = 0,\\ &\mathrm{i}\psi_\tau + \psi_{\xi\xi} + 2(|\psi|^2 + 2|\phi|^2)\psi - 2\phi^2\psi^* = 0, \end{aligned} \tag{4}$$

the last terms of which are known as the coherent coupling terms and govern the energy exchange between two axes of the fiber. For this model, the exact one- and two-soliton solutions have been constructed through the bilinear method [17]. In [18], the multi-soliton and the soliton interaction behaviours have been presented by virtue of the Darboux transformation.

In system (2), because the ratio between the parameters of self-phase modulation (SPM) and cross-phase modulation (XPM) is considered to be equal, this type of integrable coupled NLS system is physically valid only for the pulse propagation in a special kind of optical fiber system [19]. But for wavelength division multiplexing in single-model fibers and for the propagations of linearly polarized pulses in birefringent fibers, the ratio between SPM and XPM in the integrable system, e. g., system (4), has to be $1/2$ [14]. In general, the coupled NLS system (1) with arbitrary coefficients is not integrable.

As reported in [20], the partially coherent multi-soliton complexes [21, 22] propagating in photorefractive media can exhibit high nonlinearity with extremely low optical power. Since the multi-soliton complexes are special cases of the higher-order bright soliton solutions [11], the study of varying profiles of these experimentally observed multi-soliton complexes and their interesting collision behaviours will be facilitated by searching for the higher-order soliton solutions of NLS systems. Therefore, the investigation of coupled NLS equations both integrable and nonintegrable is necessary and of considerable significance for these topics [11].

In the present paper, with the aid of symbolic computation [2 – 4], we focus on constructing solutions for system (1) by employing the bilinear method [23], where the procedure can be generalized in principle to higher-order soliton solutions. The exact analytical one-, two-, and three-soliton solutions (including the singular and bright soliton ones) are obtained under two constraints, I: $\delta_2 = 0$ and $\delta_1 \neq \pm\delta_3$; II: $\delta_2 \neq 0$ and $\delta_1 \pm \delta_2 - \delta_3 = 0$, recovering the aforementioned cases, especially when the ratio between the parameters of SPM and XPM is not 1/2. Based on the obtained solutions, some main propagation and interaction properties of the solitons will be discussed. In particular, we point out that the pairwise collision [11] and partially coherent interaction [20 – 22] features also persist in system (1) as in the Manakov case.

The outline of the present paper is as follows: in Section 2, the exact analytical soliton solutions for system (1) will be given; in Section 3, propagation and interaction properties of the solitons will be investigated through graphical illustration; Section 4 will serve for a summary.

2. Soliton Solutions for System (1)

By introducing the dependent variable transformations

$$\phi = \frac{G}{F}, \quad \psi = \frac{H}{F}, \tag{5}$$

where $G(\xi,\tau)$ and $H(\xi,\tau)$ are both complex differentiable functions and $F(\xi,\tau)$ is a real one, system (1) can be decoupled into the bilinear forms

$$(\mathrm{i}D_\tau + pD_\xi^2)(G\cdot F) = 0,$$

$$(\mathrm{i}D_\tau + pD_\xi^2)(H\cdot F) = 0,$$

$$\begin{aligned}&(\delta_1|G|^2+\delta_3|H|^2)G+\delta_2H^2G^*\\&-pGD_\xi^2(F\cdot F)=0,\\&(\delta_1|H|^2+\delta_3|G|^2)H+\delta_2G^2H^*\\&-pHD_\xi^2(F\cdot F)=0.\end{aligned}\tag{6}$$

Here $D_\xi^m D_\tau^n$ is the bilinear operator [23] defined by

$$D_\xi^m D_\tau^n(a\cdot b)\equiv\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi'}\right)^m\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\tau'}\right)^n\cdot a(\xi,\tau)b(\xi',\tau')\Big|_{\xi'=\xi,\tau'=\tau}.$$

In the following, we will construct the exact analytical one-, two-, and three-soliton solutions for system (1) by solving (6).

2.1. One-Soliton Solutions

To obtain the one-soliton solutions, we expand the functions G, H, and F as

$$G=\varepsilon g_1,\quad H=\varepsilon h_1,\quad F=1+\varepsilon^2 f_2,\tag{7}$$

where ε is a formal expansion parameter that can be taken as 1 [23]. With the assumption that g_1 and h_1 have solutions of the form

$$g_1=a\mathrm{e}^\theta,\quad h_1=b\mathrm{e}^\theta,\quad \theta=k\xi+\omega\tau,\tag{8}$$

where a, b, k, ω are arbitrary nonzero complex parameters, we substitute expressions (7) into (6) and then equate to zero the coefficients of the terms with the same power of ε. The resulting set of equations can be solved recursively to obtain the dispersive relation and function f_2. Via symbolic computation, considering the values of parameters δ_1, δ_2 and δ_3, we obtain two sets of one-soliton solutions for system (1).

I-1: $\delta_2=0$ and $\delta_1\neq\pm\delta_3$. Then

$$\omega=\mathrm{i}k^2p,\quad f_2=\frac{|a|^2(\delta_1+\delta_3)}{2p(k_1+k_1^*)}\mathrm{e}^{\theta+\theta^*},\quad |a|=|b|.$$

When $p(\delta_1+\delta_3)>0$, the one-soliton solution for system (1) written in conventional form is

$$\begin{aligned}\phi&=\frac{a\mathrm{Re}(k)}{|a|}\sqrt{\frac{2p}{\delta_1+\delta_3}}\mathrm{e}^{\mathrm{i}\left\{p\tau\left[\mathrm{Re}(k)^2-\mathrm{Im}(k)^2\right]+\mathrm{Im}(k)\xi\right\}}\\&\quad\cdot\mathrm{sech}\left[\mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{|a|^2(\delta_1+\delta_3)}{8p\mathrm{Re}(k)^2}\right],\\\psi&=\frac{b}{a}\phi;\end{aligned}\tag{9}$$

when $p(\delta_1+\delta_3)<0$, then it is

$$\begin{aligned}\phi&=\frac{-a\mathrm{Re}(k)}{|a|}\sqrt{\frac{-2p}{\delta_1+\delta_3}}\mathrm{e}^{\mathrm{i}\left\{p\tau\left[\mathrm{Re}(k)^2-\mathrm{Im}(k)^2\right]+\mathrm{Im}(k)\xi\right\}}\\&\quad\cdot\mathrm{csch}\left[\mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{-|a|^2(\delta_1+\delta_3)}{8p\mathrm{Re}(k)^2}\right],\\\psi&=\frac{b}{a}\phi.\end{aligned}\tag{10}$$

II-1: $\delta_2\neq0$ and $\delta_1\pm\delta_2-\delta_3=0$. Then

$$\omega=\mathrm{i}k^2p,\quad f_2=\frac{|a|^2\delta_1(1+|\gamma|^2)}{2p(k_1+k_1^*)}\mathrm{e}^{\theta+\theta^*},\quad b=\gamma a,$$

where γ is an arbitrary constant satisfying $\frac{\gamma}{\gamma^*}=\frac{\delta_1-\delta_3}{\delta_2}$. Consequently, the one-soliton solution for system (1) written in conventional form, when $p\delta_1>0$, is

$$\begin{aligned}\phi&=\frac{a\mathrm{Re}(k)}{|a|}\sqrt{\frac{2p}{\delta_1(1+|\gamma|^2)}}\mathrm{e}^{\mathrm{i}\left\{p\tau\left[\mathrm{Re}(k)^2-\mathrm{Im}(k)^2\right]+\mathrm{Im}(k)\xi\right\}}\\&\quad\cdot\mathrm{sech}\left[\mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{|a|^2\delta_1(1+|\gamma|^2)}{8p\mathrm{Re}(k)^2}\right],\\\psi&=\gamma\phi;\end{aligned}\tag{11}$$

when $p\delta_1<0$, then it is

$$\begin{aligned}\phi&=\frac{-a\mathrm{Re}(k)}{|a|}\sqrt{\frac{-2p}{\delta_1(1+|\gamma|^2)}}\mathrm{e}^{\mathrm{i}\left\{p\tau\left[\mathrm{Re}(k)^2-\mathrm{Im}(k)^2\right]+\mathrm{Im}(k)\xi\right\}}\\&\quad\cdot\mathrm{csch}\left[\mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{-|a|^2\delta_1(1+|\gamma|^2)}{8p\mathrm{Re}(k)^2}\right],\\\psi&=\gamma\phi.\end{aligned}\tag{12}$$

It is obvious that solutions (9) and (11) are the bright solitons, while solutions (10) and (12) are singular on the lines:

$$\text{I-1}:\ \mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{-|a|^2(\delta_1+\delta_3)}{8p\mathrm{Re}(k)^2}=0,$$

$$\text{II-1}:\ \mathrm{Re}(k)\xi-2p\mathrm{Re}(k)\mathrm{Im}(k)\tau+\frac12\log\frac{-|a|^2\delta_1(1+|\gamma|^2)}{8p\mathrm{Re}(k)^2}=0.$$

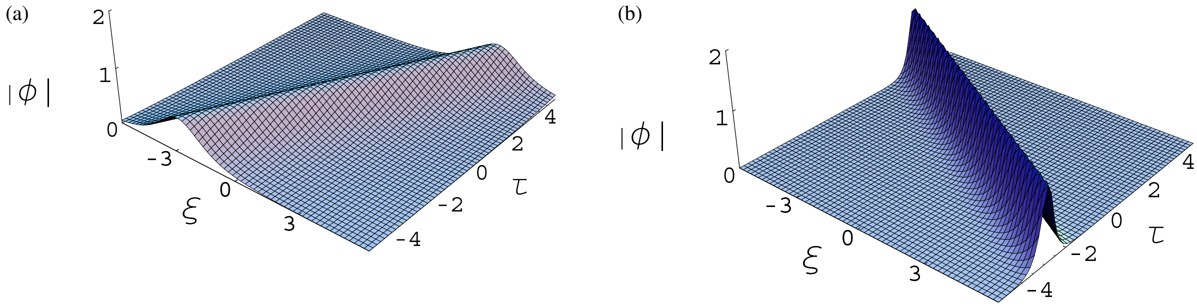


Fig. 1. Intensity plots of the bright one-soliton solution (11). The choice of the parameters is: (a) $a = 1$, $k = 1+0.3\mathrm{i}$; (b) $a = 2$, $k = 2-\mathrm{i}$; the common parameters are $\gamma = 2$ and $p = \delta_1 = 1$.

From expressions (9) and (11) we know that the amplitudes of the bright one-soliton in the first and second components are given by

$$\text{I-1}:\ A_\phi = A_\psi = \left|\mathrm{Re}(k)\sqrt{\frac{2p}{(\delta_1+\delta_3)}}\right|,$$
$$\text{II-1}:\ A_\phi = \left|\mathrm{Re}(k)\sqrt{\frac{2p}{\delta_1(1+|\gamma|^2)}}\right|,$$
$$A_\psi = \left|\gamma\mathrm{Re}(k)\sqrt{\frac{2p}{\delta_1(1+|\gamma|^2)}}\right|,$$

where the symbol A denotes the amplitude. The soliton velocities in each component are equal which can be expressed by $2p\,\mathrm{Im}(k)$. The positions of the solitons are

$$\begin{aligned}&\text{I-1}:\ \frac{1}{2\mathrm{Re}(k)}\log\frac{|a|^2(\delta_1+\delta_3)}{8p\mathrm{Re}(k)^2},\\ &\text{II-1}:\ \frac{1}{2\mathrm{Re}(k)}\log\frac{|a|^2\delta_1(1+|\gamma|^2)}{8p\mathrm{Re}(k)^2}.\end{aligned} \tag{13}$$

Based on the expressions above we can conclude that parameters p, δ_1, δ_3, γ, and the real part of k are together responsible for the soliton amplitude; p and the imaginary part of k dominate the soliton velocity; p, δ_1, δ_3, γ, modulus of a and the real part of k determine the position of soliton together. As depicted in Fig. 1, the soliton amplitude in Fig. 1b is higher than that in Fig. 1a with the increase of $\mathrm{Re}(k)$. The soliton velocity changes according to the value of $\mathrm{Im}(k)$.

2.2. Two-Soliton Solutions

For generating the two-soliton solutions, the functions G, H and F can be expanded as

$$\begin{aligned}G &= \varepsilon g_1 + \varepsilon^3 g_3,\\ H &= \varepsilon h_1 + \varepsilon^3 h_3,\\ F &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4.\end{aligned} \tag{14}$$

With the assumptions of the solutions

$$\begin{aligned}g_1 &= a_1\mathrm{e}^{\theta_1} + a_2\mathrm{e}^{\theta_2},\\ h_1 &= b_1\mathrm{e}^{\theta_1} + b_2\mathrm{e}^{\theta_2},\\ g_3 &= a_3\mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + a_4\mathrm{e}^{\theta_1+\theta_2+\theta_2^*},\\ h_3 &= b_3\mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + b_4\mathrm{e}^{\theta_1+\theta_2+\theta_2^*},\\ \theta_j &= k_j\xi + \omega_j\tau \quad (j=1,2),\end{aligned} \tag{15}$$

and manipulation as aforementioned, the dispersive relation and functions f_2 and f_4 are found to be

$$\begin{aligned}\omega_j &= \mathrm{i}k_j^2 p \quad (j=1,2),\\ f_2 &= M_{11}\mathrm{e}^{\theta_1+\theta_1^*} + M_{12}\mathrm{e}^{\theta_1+\theta_2^*}\\ &\quad + M_{21}\mathrm{e}^{\theta_2+\theta_1^*} + M_{22}\mathrm{e}^{\theta_2+\theta_2^*},\\ f_4 &= M_4\mathrm{e}^{\theta_1+\theta_1^*+\theta_2+\theta_2^*},\end{aligned}$$

where

$$\begin{aligned}a_3 &= \frac{(k_1-k_2)(a_1\kappa_{21}-a_2\kappa_{11})}{(k_1+k_1^*)(k_2+k_1^*)},\\ a_4 &= \frac{(k_2-k_1)(a_2\kappa_{12}-a_1\kappa_{22})}{(k_1+k_2^*)(k_2+k_2^*)},\\ M_{jl} &= \frac{\kappa_{jl}}{(k_j+k_l^*)} \quad (j,l=1,2),\\ M_4 &= \frac{|k_1-k_2|^2(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})}{(k_1+k_1^*)(k_2+k_2^*)|(k_1+k_2^*)|^2}.\end{aligned} \tag{16}$$

According to the values of δ_1, δ_2, and δ_3, various parameters in expressions (16) are defined as follows:

I-2: $\delta_2 = 0$ and $\delta_1 \neq \pm\delta_3$. Then

$$\kappa_{jl} = \frac{a_j a_l^*(\delta_1+\delta_3)}{2p(k_j+k_l^*)},$$
$$b_m = \gamma a_m \quad (m=1,2,3,4),$$
$$|\gamma| = 1.$$

II-2: $\delta_2 \neq 0$ and $\delta_1 \pm \delta_2 - \delta_3 = 0$. Then

$$\kappa_{jl} = \frac{a_j a_l^* \delta_1(1+|\gamma|^2)}{2p(k_j+k_l^*)},$$
$$b_m = \gamma a_m \quad (m=1,2,3,4),$$
$$\frac{\gamma}{\gamma^*} = \frac{\delta_1-\delta_3}{\delta_2}.$$

In the above cases, a_j, k_j $(j,l=1,2)$, and γ are all arbitrary nonzero complex constants.

Therefore, through the transformations (5), the two-soliton solution for system (1) can be written as

$$\begin{aligned}\phi = {}& \left(a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_3 e^{\theta_1+\theta_2+\theta_1^*} + a_4 e^{\theta_1+\theta_2+\theta_2^*}\right) \Big/ \\ & \Big(1 + M_{11}e^{\theta_1+\theta_1^*} + M_{12}e^{\theta_1+\theta_2^*} + M_{21}e^{\theta_2+\theta_1^*} \\ & + M_{22}e^{\theta_2+\theta_2^*} + M_4 e^{\theta_1+\theta_1^*+\theta_2+\theta_2^*}\Big), \\ \psi = {}& \gamma\phi.\end{aligned} \tag{17}$$

2.3. Three-Soliton Solutions

It is possible to derive three-soliton solutions with the general expressions for G, H and F as

$$\begin{aligned} G &= \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5, \\ H &= \varepsilon h_1 + \varepsilon^3 h_3 + \varepsilon^5 h_5, \\ F &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6, \end{aligned} \tag{18}$$

where

$$g_1 = a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_3 e^{\theta_3},$$
$$h_1 = b_1 e^{\theta_1} + b_2 e^{\theta_2} + b_3 e^{\theta_3},$$
$$\begin{aligned} g_3 = {}& a_{121}e^{\theta_1+\theta_2+\theta_1^*} + a_{122}e^{\theta_1+\theta_2+\theta_2^*} + a_{233}e^{\theta_2+\theta_3+\theta_3^*} \\ & + a_{232}e^{\theta_2+\theta_3+\theta_2^*} + a_{133}e^{\theta_1+\theta_3+\theta_3^*} + a_{131}e^{\theta_1+\theta_3+\theta_1^*} \\ & + a_{123}e^{\theta_1+\theta_2+\theta_3^*} + a_{231}e^{\theta_2+\theta_3+\theta_1^*} + a_{132}e^{\theta_1+\theta_3+\theta_2^*}, \end{aligned}$$
$$\begin{aligned} h_3 = {}& b_{121}e^{\theta_1+\theta_2+\theta_1^*} + b_{122}e^{\theta_1+\theta_2+\theta_2^*} + b_{233}e^{\theta_2+\theta_3+\theta_3^*} \\ & + b_{232}e^{\theta_2+\theta_3+\theta_2^*} + b_{133}e^{\theta_1+\theta_3+\theta_3^*} + b_{131}e^{\theta_1+\theta_3+\theta_1^*} \\ & + b_{123}e^{\theta_1+\theta_2+\theta_3^*} + b_{231}e^{\theta_2+\theta_3+\theta_1^*} + b_{132}e^{\theta_1+\theta_3+\theta_2^*}, \end{aligned}$$
$$\begin{aligned} g_5 = {}& a_{12}e^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*} + a_{23}e^{\theta_1+\theta_2+\theta_3+\theta_2^*+\theta_3^*} \\ & + a_{13}e^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_3^*}, \end{aligned}$$
$$\begin{aligned} h_5 = {}& b_{12}e^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*} + b_{23}e^{\theta_1+\theta_2+\theta_3+\theta_2^*+\theta_3^*} \\ & + b_{13}e^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_3^*}, \end{aligned}$$
$$\begin{aligned} f_2 = {}& M_{11}e^{\theta_1+\theta_1^*} + M_{12}e^{\theta_1+\theta_2^*} + M_{13}e^{\theta_1+\theta_3^*} \\ & + M_{21}e^{\theta_2+\theta_1^*} + M_{22}e^{\theta_2+\theta_2^*} + M_{23}e^{\theta_2+\theta_3^*} \\ & + M_{31}e^{\theta_3+\theta_1^*} + M_{32}e^{\theta_3+\theta_2^*} + M_{33}e^{\theta_3+\theta_3^*}, \end{aligned}$$
$$\begin{aligned} f_4 = {}& M_{1212}e^{\theta_1+\theta_2+\theta_1^*+\theta_2^*} + M_{1313}e^{\theta_1+\theta_3+\theta_1^*+\theta_3^*} \\ & + M_{2323}e^{\theta_2+\theta_3+\theta_2^*+\theta_3^*} + M_{1312}e^{\theta_1+\theta_3+\theta_1^*+\theta_2^*} \\ & + M_{1213}e^{\theta_1+\theta_2+\theta_1^*+\theta_3^*} + M_{2312}e^{\theta_2+\theta_3+\theta_1^*+\theta_2^*} \\ & + M_{1223}e^{\theta_1+\theta_2+\theta_2^*+\theta_3^*} + M_{1323}e^{\theta_1+\theta_3+\theta_2^*+\theta_3^*} \\ & + M_{2313}e^{\theta_2+\theta_3+\theta_1^*+\theta_3^*}, \end{aligned}$$
$$f_6 = M_6 e^{\theta_1+\theta_1^*+\theta_2+\theta_2^*+\theta_3+\theta_3^*},$$
$$\theta_j = k_j\xi + \mathrm{i}k_j^2 p\tau \quad (j=1,2,3). \tag{19}$$

By solving the resulting linear partial differential equations recursively via symbolic computation, the parameters in expressions (19) are determined to be

$$a_{mjl} = \frac{(k_m-k_j)(a_m\kappa_{jl} - a_j\kappa_{ml})}{(k_m+k_l^*)(k_j+k_l^*)},$$

$$b_{mjl} = \frac{(k_m-k_j)(b_m\kappa_{jl} - b_j\kappa_{ml})}{(k_m+k_l^*)(k_j+k_l^*)},$$

$$\begin{aligned} a_{jl} = {}& \left[(k_1-k_2)(k_2-k_3)(k_3-k_1)(k_j^*-k_l^*)\right]\Big[(k_1+k_j^*) \\ & \cdot(k_1+k_l^*)(k_2+k_j^*)(k_2+k_l^*)(k_3+k_j^*)(k_3+k_l^*)\Big]^{-1} \\ & \cdot\Big[a_3(\kappa_{1l}\kappa_{2j} - \kappa_{1j}\kappa_{2l}) - a_2(\kappa_{1l}\kappa_{3j} - \kappa_{1j}\kappa_{3l}) \\ & + a_1(\kappa_{2l}\kappa_{3j} - \kappa_{2j}\kappa_{3l})\Big], \end{aligned}$$

$$\begin{aligned} b_{jl} = {}& \left[(k_1-k_2)(k_2-k_3)(k_3-k_1)(k_j^*-k_l^*)\right]\Big[(k_1+k_j^*) \\ & \cdot(k_1+k_l^*)(k_2+k_j^*)(k_2+k_l^*)(k_3+k_j^*)(k_3+k_l^*)\Big]^{-1} \\ & \cdot\Big[b_3(\kappa_{1l}\kappa_{2j} - \kappa_{1j}\kappa_{2l}) - b_2(\kappa_{1l}\kappa_{3j} - \kappa_{1j}\kappa_{3l}) \\ & + b_1(\kappa_{2l}\kappa_{3j} - \kappa_{2j}\kappa_{3l})\Big], \end{aligned}$$

$$M_{jl} = \frac{\kappa_{jl}}{(k_j+k_l^*)},$$

$$M_{mjlq} = \frac{(k_m-k_j)(k_l^*-k_q^*)(\kappa_{ml}\kappa_{jq} - \kappa_{mq}\kappa_{jl})}{(k_m+k_l^*)(k_m+k_q^*)(k_j+k_l^*)(k_j+k_q^*)},$$

$$M_6 = \left(|k_1 - k_2|^2 |k_1 - k_3|^2 |k_2 - k_3|^2\right) \Big/ \Big[(k_1 + k_1^*)(k_2 + k_2^*)(k_3 + k_3^*)|k_1 + k_2^*|^2 |k_1 + k_3^*|^2 |k_2 + k_3^*|^2\Big] \cdot (\kappa_{12}\kappa_{23}\kappa_{31} - \kappa_{13}\kappa_{22}\kappa_{31} + \kappa_{13}\kappa_{21}\kappa_{32} - \kappa_{11}\kappa_{23}\kappa_{32} - \kappa_{12}\kappa_{21}\kappa_{33} + \kappa_{11}\kappa_{22}\kappa_{33}),$$

where $m, j, l, q = 1,2,3$, a_j and k_j are arbitrary nonzero complex constants. For different cases of δ_1, δ_2 and δ_3, the various parameters given in above expressions are defined as those in I-2 and II-2 for the two-soliton solutions.

Therefore, the three-soliton solution for system (1) can be derived as

$$\phi = \frac{g_1 + g_3 + g_5}{1 + f_2 + f_4 + f_6}, \quad \psi = \gamma\phi. \tag{20}$$

The above procedure can be straightforwardly generalized to construct the four-soliton and higher-order soliton solutions. Although the expressions will be quite lengthy with the increase of order, it can be explicitly represented in terms of exponential functions (the details are omitted here).

3. Pairwise Collisions and Partially Coherent Interactions of the Bright Solitons

According to the solutions obtained in the previous section, we will detailedly investigate the propagation and interaction characteristics of the bright solitons.

With the aid of symbolic calculations, by virtue of system (1) and its complex conjugate equations, it is easy to obtain the energy conservation law as

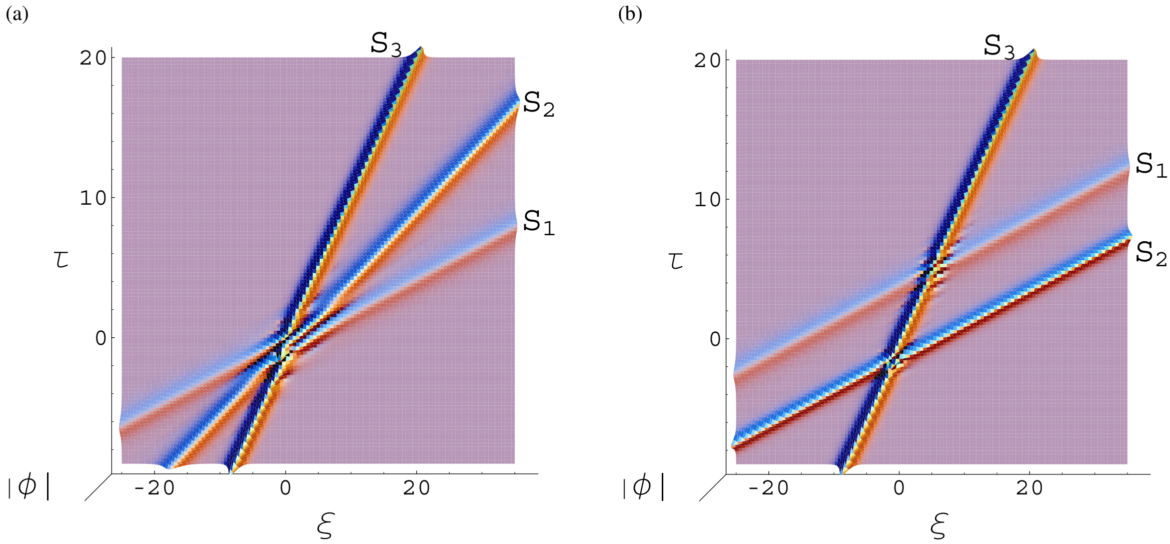
$$\mathrm{i}\frac{\partial}{\partial\tau}\left(|\phi|^2 + |\psi|^2\right) = \frac{\partial}{\partial\xi}\left[p\left(\phi\phi_\xi^* - \phi^*\phi_\xi + \psi\psi_\xi^* - \psi^*\psi_\xi\right)\right].$$

For the obtained two- and three-soliton solutions in Section 2, because the ratios of parameters a_j to b_j $(j = 1,2,3)$ are equal, the collisions of the solitons are elastic, and there is no energy exchange between the two components except for hardly visible phase shifts, whereas the interesting pairwise collision and partially coherent interaction features persist in system (1) for three- or higher-order solitons as in the Manakov case [11]. In the following, utilizing the three-soliton solution of case II, some figures will be plotted to explicitly illustrate the pairwise collisions and the partially coherent interactions.

Figure 2 displays the pairwise elastic collisions of three solitons with different amplitudes and velocities expressed via solution (20). As analyzed above, the three solitons pass through each other unaffectedly with only small phase shifts during the process of collision. Similar to the situation studied in [11], the collision is indeed a pairwise one which can be seen directly and clearly in Figure 2. We can learn from Fig. 2a that the centres of the three solitons collide with each other simultaneously at the same spot with the proper choice of parameters to ensure the propagation lines of the three solitons intersect at the same point. When we set $\mathrm{Im}(k_1) = \mathrm{Im}(k_2) \neq \mathrm{Im}(k_3)$, the corresponding solitons S_1 and S_2 can propagate in parallel, while S_3 does not. As demonstrated in Fig. 2b, S_3, after the first collision with S_2, collides with S_1. The last plot in Fig. 2 gives the perfect illustration of the pairwise collision among three solitons at three different positions. The reason why the three collisions occur in different forms can be deduced from the different choices of a_j and k_j $(j = 1,2,3)$, which affect the velocities and positions of the solitons significantly. Similar to the one-soliton case, if parameters p, δ_1, and γ are given, the velocities of the colliding solitons expressed by solution (20) are determined by the imaginary parts of k_j, and the positions are related to the modulus of a_j and the real parts of k_j $(j = 1,2,3)$.

When the relative separation distance of the solitons is small enough, a partially coherent interaction will take place, which is shown in Figure 3. Under the partially coherent interaction, the three parallel solitons superpose nonlinearly [22], thus the three-soliton complex [11,20] is formed. Figure 3 evidently shows that the profile of the complex varies periodically, and this strongly depends on the values of a_j and k_j $(j = 1,2,3)$, which determine how the solitons spread out. As in the case of the one soliton, the modulus of a_j and the real parts of k_j $(j = 1,2,3)$ influence the soliton positions (see expression (13)), and further determine the relative distances of solitons. The smaller the relative distance of solitons is, the more intense the mutual interaction will be. Consequently, the three-soliton complex will possess a rich variety of structures according to the varied propagation parameters of the single-soliton components.

In fact, a partially coherent interaction of the solitons exists not only when the solitons transmit in parallel but also when they collide. The difference between



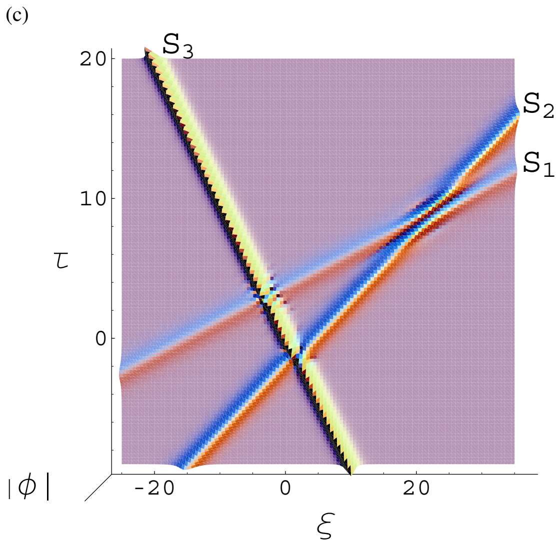


Fig. 2. Planform of pairwise elastic collisions of three solitons expressed by solution (20). The choice of the parameters is: (a) $a_1 = 0.5$, $a_2 = 1$, $k_2 = 1 + \mathrm{i}$, $k_3 = 2 + 0.5\mathrm{i}$; (b) $a_1 = 1000$, $a_2 = 0.1$, $k_2 = 1 + 2\mathrm{i}$, $k_3 = 2$; (c) $a_1 = 1000$, $a_2 = 0.1$, $k_2 = 1 + \mathrm{i}$, $k_3 = 2 - 0.5\mathrm{i}$; the common parameters are $k_1 = 0.5 + 2\mathrm{i}$, $\gamma = \sqrt{3}$, $\delta_1 = 2$, $p = 1$, and $a_3 = 2$.

the two cases is that the interaction merely occurs during the collision process when the relative separation distance of the solitons is small enough. As an example of this, Fig. 4 demonstrates the collision scenario between a single soliton and a two-soliton complex. Different from the case in Fig. 2b, the two parallel solitons merge together to form a two-soliton complex. During the collision process, as exhibited in Fig. 4b, the partially coherent interaction occurs among the three solitons, then a three-soliton complex is formed. The shape of the complex is variable and disappears after collision with the increase of relative separation distance. It is worth mentioning that such an interaction also exists in the collision process in Figure 2.

4. Summary

A generalized (1+1)-dimensional coupled nonlinear Schrödinger system with mixed nonlinear interactions has been investigated, which has potential applications in nonlinear optics and elastic solids. It is known that constructing the exact analytic three-soliton solutions is a very laborious and tedious task for NLS-type equations. However, computerized symbolic computation [2–4], a branch of artificial intelligence, is able to improve drastically the computation ability to deal with this problem exactly and algorithmically. Thus with the help of symbolic computation, the one-, two- and three-soliton solutions for system (1) have been

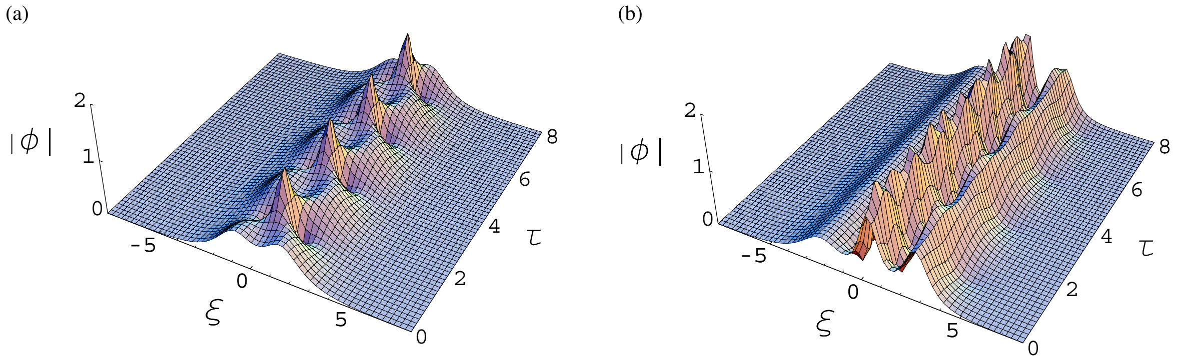
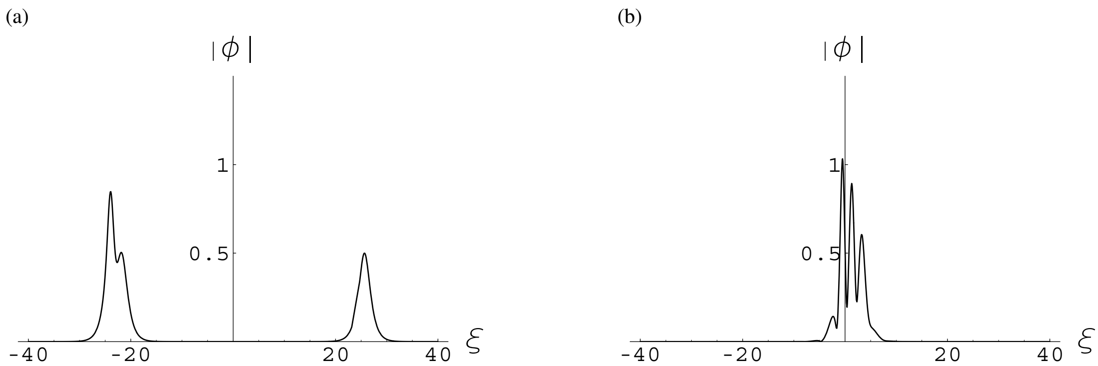


Fig. 3. Partially coherent interaction of three parallel solitons expressed by solution (20). The choice of the parameters is: (a) $a_1 = a_3 = k_3 = 1$; (b) $a_1 = 8$, $a_3 = 3$, $k_3 = 3$; the common parameters are $a_2 = 2$, $k_1 = 1$, $k_2 = 2$, $\gamma = \sqrt{3}$, $\delta_1 = 2$, and $p = 1$.



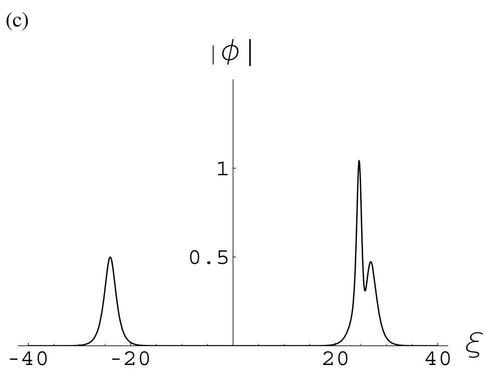


Fig. 4. Intensity profiles showing the collision scenario between a single soliton and a two-soliton complex described by solution (20), at (a) $\tau = -12$; (b) $\tau = 0$; (c) $\tau = 12$. The choice of the other parameters is: $a_1 = a_3 = 1$, $a_2 = 2$, $k_1 = 1+\mathrm{i}$, $k_2 = 2+\mathrm{i}$, $k_3 = 1-\mathrm{i}$, $\gamma = \sqrt{3}$, $\delta_1 = 2$, and $p = 1$.

derived through the bilinear method under two constraints, and this procedure can be straightforwardly generalized to construct higher-order soliton solutions. It has been given in [17], that system (1) can pass the Painlevé test when the parameters satisfy: (i) $\delta_1 = \delta_3$, $\delta_2 = 0$ or (ii) $\delta_3 = 2\delta_1$, $\delta_2 = \pm\delta_1$. System (1) under condition (i) is the celebrated Manakov type that has been investigated in many papers [9–11], while with condition (ii), e. g., system (3), it is only a special situation of case II considered in this paper, and the solutions given in [17] are consistent with ours. Furthermore, on the basis of the three-soliton solutions, the pairwise collisions and partially coherent interactions of the solitons have been analyzed through graphical illustration.

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